Generalized Weinberg-Tucker-Hammer equations in the Petiau-Duffin-Kemmer form

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Abstract

Massive and massless vector particles, in the framework of generalized Weinberg-Tucker-Hammer equations, are considered in the Petiau-Duffin-Kemmer formalism. We obtain solutions of equations in the form of matrix-dyads. The Lagrangian is defined in the matrix form and the conserved electric current and the energy-momentum tensor are obtained. The canonical quantization is performed and the propagator of massive fields is found in the formalism considered.

1 Introduction

The vector fields play very important role in the Standard Model which unifies weak and electromagnetic interactions of elementary particles. After the spontaneous breaking symmetry massless vector fields acquire masses due to the Higgs mechanism. But the existence of scalar Higgs bosons in nature is questionable and is verified at the Large Hadronic Collider. Therefore, the alternative description of massive and massless vector particles is of great theoretical interest. The second-order Proca equations, that are used in the theory of vector particles, can be represented in the form of the first-order Petiau-Duffin-Kemmer (PDK) relativistic wave equation [1], [2], [3] (see also [4]). The PDK form of equations is very convenient for different applications [5]. We have used the PDK form of equations in the Dirac-Kähler theory [6], in the field theory of multi-spin particles [7], [8], and in generalized electrodynamics [9].

In this paper, we consider generalized Weinberg-Tucker-Hammer equations (WTH) for massive and massless vector fields, introduced by Dvoeglazov [10], in the PDK formalism. We obtain solutions in the form of projection

matrix-dyads, find the conserved electric current and the energy-momentum tensor, and perform the canonical quantization in the PDK form.

The paper is organized as follows. In Sec.2, we represent generalized WTH equations for massive and massless vector fields in the form of generalized PDK equations. Solutions of the wave equation for a free fields are obtained in the form of matrix-dyads in Sec.3. We define the Lagrangian in the matrix form and find the conserved electric current and the energy-momentum tensor in Sec.4. The canonical quantization of fields is performed and the propagator is obtained in the PDK form in Sec.5. We discuss the results obtained in Sec.6. Some useful matrix relations are found in Appendix.

The Euclidean metric is used and the system of units $\hbar = c = 1$ is explored. Greek and Latin letters run $1, 2, 3, 4$ and $1, 2, 3$, respectively, and, we imply the summation in the repeated indexes.

2 Generalized WTH equations in the PDK formalism

Let us consider generalized WTH equations for vector fields, introduced by Dvoeglazov [10]:

$$\left(\gamma_{\alpha\beta}\partial_\alpha\partial_\beta + A\partial_\alpha^2 - Bm^2\right)\psi(x) = 0, \tag{1}$$

where, $\gamma_{\alpha\beta}$ are 6×6-matrices given in [11], $\partial_\mu = (\partial/\partial x_m, \partial/\partial(it))$, A, B are dimensionless parameters, and m is the mass parameter. The wave function $\psi =$column$(\mathbf{E}+i\mathbf{B}, \mathbf{E}-i\mathbf{B})$, (where $E_a = iF_{a4}$, $B_a = (1/2)\epsilon_{abc}F_{bc}$) realizes the $(1,0)\oplus(0,1)$ representation of the Lorentz group. Eq.(1) is the generalization of the Weinberg [12] and Tucker-Hammer equations [13]. At $A = 0$, $B = 1$ one arrives at the Weinberg equation, and at $A = 1$, $B = 2$, we have the Tucker-Hammer equation [10]. In the component form Eq.(1) is equivalent to the second-order equation for the antisymmetric tensor $F_{\alpha\beta}$ [10]:

$$\partial_\alpha\partial_\mu F_{\mu\beta} - \partial_\beta\partial_\mu F_{\mu\alpha} + \frac{A-1}{2}\partial_\mu^2 F_{\alpha\beta} - \frac{B}{2}m^2 F_{\alpha\beta} = 0, \tag{2}$$

We study Eq.(2) in the PDK form as for the massive case, $m \neq 0$, as well as for the massless case, $m = 0$.

2.1 Massive vector fields

We introduce the 10-component wave function

$$\Psi(x) = \{\psi_A(x)\} = \left(\begin{array}{c} \psi_\mu(x) \\ \psi_{[\mu\nu]}(x) \end{array} \right) = \left(\begin{array}{c} \frac{1}{m}\partial_\nu F_{\mu\nu}(x) \\ F_{\mu\nu}(x) \end{array} \right) \qquad (A = \mu, [\mu\nu]). \quad (3)$$

With the help of the elements of the entire matrix algebra $\varepsilon^{A,B}$, with products and matrix elements

$$\left(\varepsilon^{M,N}\right)_{AB} = \delta_{MA}\delta_{NB}, \qquad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \quad (4)$$

where $A, B, M, N = \mu, [\mu\nu]$, Eq.(2) can be represented in the form

$$\left[\partial_\mu \left(\varepsilon^{\nu,[\nu\mu]} + \varepsilon^{[\nu\mu],\nu}\right) + m\varepsilon^{\mu,\mu} \right. \\ \left. + \left(\frac{1-A}{2m}\partial_\mu^2 + \frac{B}{2}m\right)\frac{1}{2}\varepsilon^{[\nu\mu],[\nu\mu]}\right]_{AB} \Psi_B(x) = 0. \quad (5)$$

We imply a summation over all repeated indices. Defining the 10×10 matrices

$$\beta_\mu = \varepsilon^{\nu,[\nu\mu]} + \varepsilon^{[\nu\mu],\nu}, \quad \overline{P} = \varepsilon^{\mu,\mu}, \quad P = \frac{1}{2}\varepsilon^{[\nu\mu],[\nu\mu]}, \quad (6)$$

Eq.(5) takes the PDK matrix form

$$\left[\beta_\mu\partial_\mu + m\overline{P} + P\left(\frac{1-A}{2m}\partial_\mu^2 + \frac{B}{2}m\right)\right]\Psi(x) = 0, \quad (7)$$

where the matrices β_μ are Hermitian matrices, $\beta_\mu^+ = \beta_\mu$. Eq.(7) is the generalized PDK equation. The projection operator $\overline{P} = \overline{P}^+$ extracts the four-dimensional vector subspace (ψ_μ) of the wave function Ψ, and the projection operator $P = P^+$ extracts the six-dimensional tensor subspace corresponding to the $\psi_{[\mu\nu]}$. The matrices β_μ obey the PDK algebra

$$\beta_\mu\beta_\nu\beta_\alpha + \beta_\alpha\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\alpha + \delta_{\alpha\nu}\beta_\mu, \quad (8)$$

and matrices $\overline{P}$, P are projection matrices

$$\overline{P}^2 = \overline{P}, \quad P^2 = P, \quad \overline{P} + P = 1, \quad \overline{P}P = P\overline{P} = 0, \quad (9)$$

$$\beta_\mu \overline{P} + \overline{P}\beta_\mu = \beta_\mu, \quad \beta_\mu P + P\beta_\mu = \beta_\mu.$$

The Lorentz group generators in the 10-dimension representation space are given by [7], [8]

$$J_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu = \varepsilon^{[\lambda\mu],[\lambda\nu]} + \varepsilon^{\mu,\nu} - \varepsilon^{[\lambda\nu],[\lambda\mu]} - \varepsilon^{\nu,\mu}, \tag{10}$$

and obey the commutation relations

$$\begin{gathered}
[J_{\rho\sigma}, J_{\mu\nu}] = \delta_{\sigma\mu}J_{\rho\nu} + \delta_{\rho\nu}J_{\sigma\mu} - \delta_{\rho\mu}J_{\sigma\nu} - \delta_{\sigma\nu}J_{\rho\mu}, \\
[\beta_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu, \quad \left[\overline{P}, J_{\mu\nu}\right] = 0, \quad [P, J_{\mu\nu}] = 0.
\end{gathered} \tag{11}$$

Eq.(7) is form-invariant because of Eq.(11). The Lorentz-invariant is $\overline{\Psi}\Psi = \Psi^+\eta\Psi$, where Ψ^+ is the Hermitian-conjugated wave function, and the Hermitianizing matrix η is

$$\eta = \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2}\varepsilon^{[mn],[mn]}. \tag{12}$$

The η is the Hermitian matrix, $\eta^+ = \eta$, and obeys the relations: $\eta\beta_m = -\beta_m\eta$ (m=1,2,3), $\eta\beta_4 = \beta_4\eta$. With the help of these relations, one finds the "conjugated" equation

$$\overline{\Psi}(x)\left[\beta_\mu \overleftarrow{\partial}_\mu - m\overline{P} - \left(\overleftarrow{\partial}_\mu^2 \frac{1-A}{2m} + \frac{B}{2}m\right)P\right] = 0, \tag{13}$$

where $\overline{\Psi} = (\psi^*_\mu, -\psi^*_{[\mu\nu]})$ and the complex conjugation $*$ does not act on the metric imaginary unit i of fourth components of the wave function, $\psi^*_\mu = (\psi^*_m, i\psi^*_0)$, and so on.

2.2 Massless vector fields

To get the massless WTH equation, we put $m = 0$ in Eq.(2) and arrive at

$$\partial_\alpha\partial_\mu F_{\mu\beta} - \partial_\beta\partial_\mu F_{\mu\alpha} + \frac{A-1}{2}\partial_\mu^2 F_{\alpha\beta} = 0. \tag{14}$$

In the same manner as for the massive case, we introduce the 10-component wave function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi_\mu(x) \\ \psi_{[\mu\nu]}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{\kappa}\partial_\nu F_{\mu\nu}(x) \\ F_{\mu\nu}(x) \end{pmatrix}, \tag{15}$$

where κ is the new parameter with the dimension of the mass. We need the dimensional parameter to have the components of the wave function $\Psi(x)$ with the same dimension. Physical values do not depend on this parameter, and κ is absorbed by the renormalization of fields [9]. As a result, Eq.(14) takes the form similar to Eq.(7) for massive fields

$$\left[\beta_\mu \partial_\mu + \kappa \overline{P} + P\frac{C}{\kappa}\partial_\mu^2\right]\Psi(x) = 0, \tag{16}$$

where C is an arbitrary parameter ($C = (A-1)/2$). Thus, the equation of motion and other equations for the massless fields can be obtained from the massive case by the replacements $m \to \kappa$, $B = 0$. Therefore, in the further consideration, we write down only the expressions for the massive case implying that they are valid also for the massless fields after the replacements $m \to \kappa$, $B = 0$. Eq.(14), (16) represent the modified Maxwell equations.

3 Solutions of the matrix equation

In the momentum space, for the positive ($+p$) and negative ($-p$) energies, Eq.(7) reads

$$\left(\pm i\widehat{p} + m\overline{P} + \lambda P\right)\Psi(\pm p) = 0, \tag{17}$$

where $\widehat{p} = \beta_\mu p_\mu$, the four-momentum is $p_\mu = (\mathbf{p}, ip_0)$, and

$$\lambda = \frac{A-1}{2m}p^2 + \frac{B}{2}m, \tag{18}$$

$p^2 = \mathbf{p}^2 - p_0^2$. The matrix of Eq.(17)

$$\Lambda_\pm = \pm i\widehat{p} + m\overline{P} + \lambda P, \tag{19}$$

obeys the minimal polynomial (see Eq.(A3) in Appendix):

$$(\Lambda_\pm - m)(\Lambda_\pm - \lambda)\left[(\Lambda_\pm - m)(\Lambda_\pm - \lambda) + p^2\right] = 0. \tag{20}$$

There exist non-trivial solutions of Eq.(17) if $\det \Lambda_\pm = 0$ or the eigenvalue of the $\Lambda_\pm$ is equal to zero. If $\lambda \neq 0$, this requirement leads to the dispersion relation

$$p^2 + \lambda m = 0. \tag{21}$$

It follows from Eq.(20) that there are also other eigenvalues of the matrix $\Lambda_\pm$: m and λ. Taking into account Eq.(18), one obtains from Eq.(21)

$$p^2 + \frac{m^2 B}{A+1} = 0. \tag{22}$$

Thus, the mass of the vector field is

$$M = m\sqrt{\frac{B}{A+1}}. \tag{23}$$

Another case which leads to $\det\Lambda_\pm = 0$, is $\lambda = 0$. Then Eq.(18) results at $\lambda = 0$ to the physical mass

$$M' = m\sqrt{\frac{B}{A-1}}. \tag{24}$$

The formulas (23), (24) are in the agreement with results obtained by Dvoeglazov [10]. From Eq.(23), we find the restriction on the parameters A and B: $B/(A+1) \geq 0$, and in the case of Eq.(24), $B/(A-1) \geq 0$. It follows from Eq.(23) that for Weinberg [12] ($A = 0$, $B = 1$) and Tucker-Hammer [13] ($A = 1$, $B = 2$) equations, we have the same physical mass of vector particles, $M = m$. The case (24) is not realized for Tucker-Hammer equations [13] ($A = 1$, $B = 2$, $\lambda \neq 0$). Therefore, we imply that $\lambda \neq 0$ and the physical masses of vector particles are given by Eq.(23). On-shell, when Eq.(22) is valid, and $m \neq \lambda$ the minimal polynomial equation (20) reduces to

$$\Lambda_\pm \left(\Lambda_\pm - \lambda - m\right)\left(\Lambda_\pm - m\right)\left(\Lambda_\pm - \lambda\right) = 0. \tag{25}$$

Eq.(25) allows us to obtain solutions of Eq.(17), $\Lambda\Psi(p) = 0$, in the form of the projection matrix [14]. Indeed, the matrix

$$\begin{aligned}\Pi_\pm &= N\left(\Lambda_\pm - \lambda - m\right)\left(\Lambda_\pm - m\right)\left(\Lambda_\pm - \lambda\right) \\ &= \mp N i\widehat{p}\left(\pm i m\widehat{p}P \pm i\lambda\widehat{p}\overline{P} - \lambda m\right),\end{aligned} \tag{26}$$

where N is the normalization constant, obeys the equation $\Lambda_\pm\Pi_\pm = 0$. It means that every column of the matrix $\Pi_\pm$ is the solution to Eq.(17). The requirement that the $\Pi_\pm$ is the projection matrix

$$\Pi_\pm^2 = \Pi_\pm, \tag{27}$$

gives the value

$$N=-\frac{1}{\lambda m(\lambda+m)}. \tag{28}$$

Eq.(28) can be verified with the help of the minimal Eq.(25). It should be noted that Eq.(25) represents the minimal polynomial for the non-degenerate case $m\neq\lambda$ (see Eq.(A3) in Appendix). But the case $m=\lambda$ is realized as for the Weinberg equation ($A=0$, $B=1$) as well as for the Tucker-Hammer equation ($A=1$, $B=2$). This follows from Eq.(18),(23). For this special case, $m=\lambda$, one should explore the minimal polynomial equation on-shell (see Eq.(A4) in Appendix)

$$\Lambda_\pm\left(\Lambda_\pm-m\right)\left(\Lambda_\pm-2m\right)=0. \tag{29}$$

which corresponds to the PDK equation for spin-1. Indeed, if $m=\lambda$, Eq.(17) becomes

$$\left(\pm i\widehat{p}+m\right)\Psi(\pm p)=0, \tag{30}$$

as $\overline{P}+P=1$. But Eq.(30) is the PDK equation for vector fields and the solution in the form of the matrix-dyad to this equation is well known [14]. We find the projection operator from Eq.(29)

$$\Pi_\pm^{PDK}=\frac{1}{2m^2}\left(\Lambda_\pm^2-3m\Lambda_\pm+2m^2\right)=\frac{\pm i\widehat{p}\left(\pm i\widehat{p}-m\right)}{2m^2}. \tag{31}$$

Eq.(26) converts to Eq.(31) at $m=\lambda$.

The projection operators extracting spin projections ±1 and 0 are given by [14],

$$S_{(\pm1)}=\frac{1}{2}\sigma_p\left(\sigma_p\pm1\right),\qquad S_{(0)}=1-\sigma_p^2. \tag{32}$$

where the spin operator is

$$\sigma_p=-\frac{i}{2\mid\mathbf{p}\mid}\epsilon_{abc}p_aJ_{bc}=-\frac{i}{\mid\mathbf{p}\mid}\epsilon_{abc}p_a\beta_b\beta_c \tag{33}$$

and commutes with the matrix of Eq.(17) $\Lambda_\pm$: $[\Lambda_\pm,\sigma_p]=0$. The projection operators satisfy the relations: $S^2_{(\pm1)}=S_{(\pm1)}$, $S_{(\pm1)}S_{(0)}=0$, $S^2_{(0)}=S_{(0)}$. The operators (32) commute with the mass projection operator (26) and, as a result, we obtain projection operators extracting solutions to Eq.(17) for spin projections ±1, 0 (for states of particles with the mass m) in the form of matrix-dyads

$$\Pi_\pm S_{(\pm1)}=\Psi_{\pm1}\cdot\overline{\Psi}_{\pm1},\qquad \Pi_\pm S_{(0)}=\Psi_0\cdot\overline{\Psi}_0. \tag{34}$$

The matrix elements of the matrix-dyad is $(\Psi \cdot \overline{\Psi})_{AB} = \Psi_A \overline{\Psi}_B$. For the case of the Weinberg and Tucker-Hammer equations, one has to make the replacement in Eq.(34) $\Pi_\pm \to \Pi_\pm^{PDK}$. Eq.(34) can be used for different electrodynamics calculations of processes with vector particles in the framework of generalized WTH model.

For the case of massless WTH equation, the parameter on-shell is $\lambda = 0$ ($p^2 = 0$). Then the operator of the equation becomes

$$\Lambda_\pm = i\widehat{p} + \kappa \overline{P}, \tag{35}$$

and obeys the minimal matrix equation on-shell (see Eq.(A6) in Appendix) as follows:

$$\Lambda_\pm^2 \left(\Lambda_\pm - \kappa\right)^2 = 0. \tag{36}$$

As pointed in [15], in this case there are difficulties to obtain the solutions in the form of projection operators because the zero eigenvalue of the operator $\Lambda_\pm$ is degenerated. In the case of Maxwell equations such difficulty was overcome in [9] by using the general gauge and adding a scalar field.

4 Lagrangian and conserved currents

The Lagrangian of generalized WTH model in the PDK formalism is given by

$$\begin{aligned} \mathcal{L} = -\frac{1}{2}\overline{\Psi}(x)\beta_\mu \left(\partial_\mu - \overleftarrow{\partial}_\mu\right)\Psi(x) - m\overline{\Psi}(x)\left(\overline{P} + \frac{B}{2}P\right)\Psi(x) \\ + \frac{1-A}{2m}\left(\partial_\mu \overline{\Psi}(x)\right) P \left(\partial_\mu \Psi(x)\right). \end{aligned} \tag{37}$$

One may construct another Lagrangian by adding the four-divergence to Eq.(37) that does not change equations of motion. It can be verified that Euler-Lagrange equations following from Eq.(37) give the equations of motion (7),(13). We obtain also the Lagrangian (37) in terms of fields ψ_A

$$\mathcal{L} = \frac{1}{2}\Big[\psi^*_{[\rho\mu]}\partial_\mu\psi_\rho - \psi^*_\rho\partial_\mu\psi_{[\rho\mu]} - m\psi^*_\mu\psi_\mu \tag{38}$$

$$+\frac{mB}{2}\psi^*_{[\rho\mu]}\psi_{[\rho\mu]}-\frac{1-A}{2m}\left(\partial_\nu\psi^*_{[\rho\mu]}\right)\left(\partial_\nu\psi_{[\rho\mu]}\right)\Big]+c.c.,$$

where the $c.c.$ is the complex conjugated expression. The electric current density is given by [16]

$$j_\mu(x)=i\left(\overline{\Psi}(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\Psi}(x)\right)}-\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi(x)\right)}\Psi(x)\right). \tag{39}$$

Replacing Eq.(37) into Eq.(39), we obtain the electric current density

$$j_\mu(x)=i\overline{\Psi}(x)\beta_\mu\Psi(x)+\frac{1-A}{2m}\left[\overline{\Psi}(x)P\partial_\mu\Psi(x)-\left(\partial_\mu\overline{\Psi}(x)\right)P\Psi(x)\right]. \tag{40}$$

One may check with the help of equations of motion (7),(13) that the electric current is conserved, $\partial_\mu j_\mu(x)=0$. The second term in Eq.(40) vanishes at $A=1$, corresponding to the Tucker-Hammer equation, and $j_\mu(x)$ takes the form similar to the Dirac theory. Using the general expression for the canonical energy-momentum tensor [16]

$$T_{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi(x)\right)}\partial_\nu\Psi(x)+\partial_\nu\overline{\Psi}(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\Psi}(x)\right)}-\delta_{\mu\nu}\mathcal{L}, \tag{41}$$

we obtain from the Lagrangian (37)

$$\begin{gathered}T_{\mu\nu}=\frac{1}{2}\left(\partial_\nu\overline{\Psi}(x)\right)\beta_\mu\Psi(x)-\frac{1}{2}\overline{\Psi}(x)\beta_\mu\partial_\nu\Psi(x)\\+\frac{1-A}{2m}\left[\left(\partial_\mu\partial_\nu\overline{\Psi}(x)\right)P\Psi(x)+\overline{\Psi}(x)P\partial_\mu\partial_\nu\Psi(x)\right]-\delta_{\mu\nu}\mathcal{L}.\end{gathered} \tag{42}$$

One may verify that the canonical energy-momentum tensor is conserved tensor, $\partial_\mu T_{\mu\nu}=0$. At $A=1$ Eq.(37),(42) are simplified. It should be noted that the Lagrangian $\mathcal{L}$, Eq.(37), does not vanish if $A\neq 1$ for fields satisfying the equations of motion. It is easy to write down expressions for j_μ and $T_{\mu\nu}$ in the component form exploring Eq.(3),(4),(6).

5 The canonical quantization

Solutions to Eq.(7) with definite energy and momentum in the form of plane waves are given by

$$\Psi_s^{(\pm)}(x)=\sqrt{\frac{m+\lambda}{2p_0V}}\Psi_s(\pm p)\exp(\pm ipx), \tag{43}$$

where V is the normalization volume, $p^2 = \mathbf{p}^2 - p_0^2 = -m^2$, and s is the spin index ($s = \pm 1, 0$). The 10-dimensional function $\Psi_s(\pm p)$ obeys Eq.(17). We use the normalization conditions

$$\int_V \overline{\Psi}_s^{(\pm)}(x)\beta_4\Psi_{s'}^{(\pm)}(x)d^3x = \pm\delta_{ss'}, \qquad \int_V \overline{\Psi}_s^{(\pm)}(x)\beta_4\Psi_{s'}^{(\mp)}(x)d^3x = 0, \tag{44}$$

where $\overline{\Psi}_s^{(\pm)}(x) = \left(\Psi_s^{(\pm)}(x)\right)^+\eta$. Eq.(44) correspond to the normalization on the charge. In the second quantized theory, the field operators are represented by

$$\begin{aligned}\Psi(x) &= \sum_{p,s}\left[a_{p,s}\Psi_s^{(+)}(x) + b_{p,s}^+\Psi_s^{(-)}(x)\right],\\ \overline{\Psi}(x) &= \sum_{p,s}\left[a_{p,s}^+\overline{\Psi}_s^{(+)}(x) + b_{p,s}\overline{\Psi}_s^{(-)}(x)\right],\end{aligned} \tag{45}$$

where the positive and negative parts of the wave function are given by Eq.(43). The creation and annihilation operators of particles, $a_{p,s}^+$, $a_{p,s}$, and antiparticles, $b_{p,s}^+$, $b_{p,s}$, obey the commutation relations as follows:

$$\begin{aligned}&[a_{p,s}, a_{p',s'}^+] = \delta_{ss'}\delta_{pp'}, \quad [a_{p,s}, a_{p',s'}] = [a_{p,s}^+, a_{p',s'}^+] = 0,\\ &[b_{p,s}, b_{p',s'}^+] = \delta_{ss'}\delta_{pp'}, \quad [b_{p,s}, b_{p',s'}] = [b_{p,s}^+, b_{p',s'}^+] = 0,\\ &[a_{p,s}, b_{p',s'}] = [a_{p,s}, b_{p',s'}^+] = [a_{p,s}^+, b_{p',s'}] = [a_{p,s}^+, b_{p',s'}^+] = 0.\end{aligned} \tag{46}$$

We obtain the commutation relations for different times from Eq.(43)-(46)

$$\begin{gathered}\left[\Psi_M(x), \Psi_N(x')\right] = \left[\overline{\Psi}_M(x), \overline{\Psi}_N(x')\right] = 0, \quad \left[\Psi_M(x), \overline{\Psi}_N(x')\right] = N_{MN}(x,x'),\\ N_{MN}(x,x') = N_{MN}^+(x,x') - N_{MN}^-(x,x'),\\ N_{MN}^+(x,x') = \sum_{p,s}\left(\Psi_s^{(+)}(x)\right)_M\left(\overline{\Psi}_s^{(+)}(x')\right)_N,\\ N_{MN}^-(x,x') = \sum_{p,s}\left(\Psi_s^{(-)}(x)\right)_M\left(\overline{\Psi}_s^{(-)}(x')\right)_N.\end{gathered} \tag{47}$$

One finds from Eq.(43),(47)

$$N_{MN}^{\pm}(x,x') = \sum_{p,s}\frac{m+\lambda}{2p_0V}\left(\Psi_s(\pm p)\right)_M\left(\overline{\Psi}_s(\pm p)\right)_N\exp[\pm ip(x-x')]. \tag{48}$$

It follows from Eq.(32) that the equation

$$S_{(+1)} + S_{(-1)} + S_{(0)} = 1, \tag{49}$$

holds, and as a result, we obtain from Eq.(34)

$$\sum_s \left(\Psi_s(\pm p)\right) \cdot \left(\overline{\Psi}_s(\pm p)\right) = \Pi_\pm = \frac{\pm i\widehat{p}\left(\pm im\widehat{p}P \pm i\lambda\widehat{p}\overline{P} - \lambda m\right)}{\lambda m\left(m+\lambda\right)}. \tag{50}$$

Taking into account Eq.(50), one finds from Eq.(48):

$$\begin{aligned} N^{\pm}_{MN}(x,x') &= \sum_p \frac{1}{2p_0 V}\left(\frac{\pm i\widehat{p}\left(\pm im\widehat{p}P \pm i\lambda\widehat{p}\overline{P} - \lambda m\right)}{\lambda m}\right)_{MN} \exp[\pm ip(x-x')] \\ &= \left(\frac{\beta_\mu\partial_\mu\left(m\beta_\mu P\partial_\mu + \lambda\beta_\mu\overline{P}\partial_\mu - \lambda m\right)}{\lambda m}\right)_{MN} \Delta_\pm(x-x'), \end{aligned} \tag{51}$$

where the singular functions are given by [16]

$$\Delta_+(x) = \sum_p \frac{1}{2p_0 V}\exp(ipx), \qquad \Delta_-(x) = \sum_p \frac{1}{2p_0 V}\exp(-ipx). \tag{52}$$

It follows from Eq.(18) that the λ (at $A \neq 1$) is the operator because in the configuration space $p^2 = -\partial_\mu^2$. With the help of the function [16]

$$\Delta_0(x) = i\left(\Delta_+(x) - \Delta_-(x)\right), \tag{53}$$

from Eq.(47),(51),(53), we obtain

$$N_{MN}(x,x') = -i\left(\frac{\beta_\mu\partial_\mu\left(m\beta_\mu P\partial_\mu + \lambda\beta_\mu\overline{P}\partial_\mu - \lambda m\right)}{\lambda m}\right)_{MN} \Delta_0(x-x'). \tag{54}$$

It should be noted that the function $\Delta_0(x)$ vanishes when $x^2 = \mathbf{x}^2 - t^2 > 0$ [16]. Using Eq.(8), one finds the relation

$$\left(\beta_\mu\partial_\mu + m\overline{P} + \lambda P\right)\frac{\beta_\mu\partial_\mu\left(m\beta_\mu P\partial_\mu + \lambda\beta_\mu\overline{P}\partial_\mu - \lambda m\right)}{\lambda m} \tag{55}$$

$$= \frac{\left(m\overline{P} + \lambda P\right)\beta_\mu \partial_\mu}{\lambda m}\left(\partial_\mu^2 - \lambda m\right).$$

Taking into account the equation [16] $\left(\partial_\mu^2 - \lambda m\right)\Delta_\pm(x) = 0$, and Eq.(51), from Eq.(55), we arrive at

$$\left(\beta_\mu \partial_\mu + m\overline{P} + \lambda P\right) N^\pm(x, x') = 0. \tag{56}$$

For bosonic fields the chronological product of two operators is defined as [16]

$$T\Psi_M(x_1)\overline{\Psi}_N(x_2) = \left\{ \begin{array}{ll} \Psi_M(x_1)\overline{\Psi}_N(x_2), & t_1 > t_2, \\ \overline{\Psi}_N(x_2)\Psi_M(x_1), & t_2 > t_1 \end{array} \right\}.$$

Then the vacuum expectation of chronological pairing of operators (the propagator) becomes

$$\begin{gathered} \langle T\Psi_M(x)\overline{\Psi}_N(y)\rangle_0 = N^c_{MN}(x-y) \\ = \theta\left(x_0 - y_0\right) N^+_{MN}(x-y) + \theta\left(y_0 - x_0\right) N^-_{MN}(x-y), \end{gathered} \tag{57}$$

where $\theta(x)$ being the theta-function. We obtain from Eq.(57):

$$\langle T\Psi(x)\cdot\overline{\Psi}(y)\rangle_0 = \frac{\beta_\mu\partial_\mu\left(m\beta_\mu P\partial_\mu + \lambda\beta_\mu \overline{P}\partial_\mu - \lambda m\right)}{\lambda m}\Delta_c(x-y), \tag{58}$$

where

$$\Delta_c(x-y) = \theta\left(x_0 - y_0\right)\Delta_+(x-y) + \theta\left(y_0 - x_0\right)\Delta_-(x-y). \tag{59}$$

Using the equation [16] $\left(\partial_\mu^2 - \lambda m\right)\Delta_c(x) = i\delta(x)$, one finds from Eq.(55),(58)

$$\left(\beta_\mu\partial_\mu + m\overline{P} + \lambda P\right)\langle T\Psi(x)\cdot\overline{\Psi}(y)\rangle_0 = i\frac{m\overline{P} + \lambda P}{\lambda m}\beta_\mu\partial_\mu\delta(x-y). \tag{60}$$

At $A \neq 1$ ($m \neq \lambda$), we have non-local operators in Eq.(58),(60) because the variable $p^2 = -\partial_\mu^2$ enters the λ. Expressions for PDK theory follow from Eq.(58),(60) at $m = \lambda$ ($A = 1$, $B = 2$), and as a result, there are only local operators.

6 Discussion

We have considered the generalized WTH equations, that describe vector fields in the PDK formalism, Eq.(7). Eq.(7) is valid for any parameters A and B. For the Tucker-Hammer equation at $A = 1$, $B = 2$ Eq.(7) reduces to PDK equation. For the Weinberg equation, at $A = 0$, $B = 1$, we obtain from Eq.(7) the generalized PDK equation which is more complicated because it contains the second derivatives. To consider solutions to the wave equation (7) for particles in external electromagnetic fields, one has to make the replacement $\partial_\mu \to \partial_\mu - ieA_\mu$ (A_μ is the vector-potential of an electromagnetic field). Then solutions to the Weinberg equation will be different compared to solutions of the PDK equation. Therefore, physical observable also will be different. The equation for the massless particles, Eq.(16), is also different (at $C \neq 0$, $A \neq 1$) comparing with the Maxwell equations in the matrix form [15], [9].

Solutions of equations obtained in the form of matrix-dyads allow us to make calculations of processes in the quantum theory of vector particles, described by generalized WTH equations, in the simple manner. From the generalized PDK Lagrangian, we have found the conserved electric current and energy-momentum tensor in the matrix PDK form that are convenient for analyzing. Quantization of fields and the propagator obtained make it possible to use the perturbation theory for different quantum calculations.

7 Appendix

From Eq.(9),(19), we obtain

$$\Lambda_\pm^2 = -\widehat{p}^2 + (m+\lambda)\,\Lambda_\pm - \lambda m, \tag{A1}$$

$$\Lambda_\pm^3 - (m+\lambda)\,\Lambda_\pm^2 + \lambda m \Lambda_\pm = \mp i p^2 \widehat{p} - m\widehat{p}^2\overline{P} - \lambda \widehat{p}^2 P, \tag{A2}$$

where

$$\widehat{p}^2\overline{P} = \overline{P}\widehat{p}^2 = p^2\varepsilon^{\mu,\mu} - p_\mu p_\nu \varepsilon^{\nu,\mu}, \qquad \widehat{p}^2 P = P\widehat{p}^2 = p_\mu p_\nu \varepsilon^{[\lambda\mu],[\lambda\nu]}.$$

Squaring Eq.(A1) and using the relation $\widehat{p}^3 = p^2\widehat{p}$, after some algebraic manipulations, we find the minimal polynomial of the matrix $\Lambda_\pm$

$$(\Lambda_\pm - m)\,(\Lambda_\pm - \lambda)\left[(\Lambda_\pm - m)\,(\Lambda_\pm - \lambda) + p^2\right] = 0. \tag{A3}$$

On-shell, one has to put $p^2 = -\lambda m$ in Eq.(A3). Eq.(A3) holds for any m and λ. At a particular case, at $m = \lambda$ the equation is simplified and becomes

$$\left(\Lambda_\pm - m\right)\left[\left(\Lambda_\pm - m\right)^2 + p^2\right] = 0. \qquad (A4)$$

It should be noted that if we put $m = \lambda$ in Eq.(A3), we get Eq.(A4) multiplied by the factor $(\Lambda_\pm - m)$, but the real minimal equation is (A4). For the massless case the matrix (35) obeys the equation as follows:

$$\Lambda_\pm\left(\Lambda_\pm - \kappa\right)\left[\Lambda_\pm\left(\Lambda_\pm - \kappa\right) + p^2\right] = 0, \qquad (A5)$$

and on-shell $p^2 = 0$.

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